Calculus Review Highlights[†]

This document highlights some of the results from differential, integral and vector calculus that are useful for multivariable calculus. Note that in many cases the descriptions are somewhat brief; if this is the case please consult other sources for more details.

1. Differential calculus

1.1. Differentials

The tangent line approximation to the function $y = f(x)$ at the point $y_0 = f(x_0)$ is

$$y - y_0 = f'(x_0)(x - x_0)\,.$$

If the deviation $\Delta x = x - x_0$ is not too large, then this gives a reasonable approximation to the original function. Writing $\Delta y = y - y_0$, we have

$$\Delta y \approx f'(x_0)\Delta x\,.$$

In the limit of *infinitesimal* changes dx and dy (also called *differentials,* the error in this approximation goes to zero, and one has

$$dy = f'(x)dx\,.$$

Differentials are useful when working with small quantities. Since an integral is essentially the sum of an infinite number of infinitesimally small quantities, differentials are useful when setting up integrals of various types.

2. Integral calculus

The basic idea of evaluating an integral, of course, is to manipulate the integrand in such a way that it is recognizable as the derivative of some known function.

2.1. Basic integrals

Here are some elementary integrals that one gets from derivatives from elementary functions.

1. $\displaystyle\int u^n\,du = \frac{1}{n+1}u^{n+1} + C \quad (n \neq -1)$

2. $\displaystyle\int \frac{du}{u} = \ln|u| + C$

3. $\displaystyle\int e^u\,du = e^u + C$

4. $\displaystyle\int \cos u\,du = \sin u + C$

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5. $\displaystyle\int \sin u\, du = -\cos u + C$

6. $\displaystyle\int \sec^2 u\, du = \tan u + C$

7. $\displaystyle\int \csc^2 u\, du = -\cot u + C$

8. $\displaystyle\int \sec u \tan u\, du = \sec u + C$

9. $\displaystyle\int \csc u \cot u\, du = \csc u + C$

10. $\displaystyle\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$

11. $\displaystyle\int \frac{du}{1+u^2} = \tan^{-1} u + C$

The last two integrals come from working out the derivatives of inverse trig functions; such derivatives are easily found by using *implicit differentiation.*

2.2. Simple substitutions

If one has an integral

$$\int f(x)\, dx$$

And one can write $f(x)$ in the form $f(x) = G'(h(x))\, h'(x)$, then using the chain rule in reverse allows evaluation of this integral:

$$\int f(x)\, dx = \int G'(h(x))\, h'(x)\, dx = \int \frac{d}{dx}\left[G'(h(x))\right] dx = G(h(x)) + C\,.$$

Actually, it's a bit too much to be able to see all at once how to break up $f(x)$ in this way; usually, one is lucky to see that $f(x) = g(h(x))h'(x)$ without knowing what function $g(x)$ is the derivative of.

In this case, we can eliminate $h(x)$ from the above by using the *substitution* $u = h(x)$. Then we have $du = h'(x)\, dx$ and

$$\int f(x)\, dx = \int g(h(x))h'(x)\, dx = \int g(u)\, du\,.$$

At this point we can deal with the function $g(u)$; if we know that $g(u) = G'(u)$ the remaining integral can be done.

In addition, we don't have to break up the integrand into a product in order to make a change of variable: if we let $u = u(x)$, then $du = u'(x)\, dx$ and $dx = du/u'(x)$. Then

$$\int f(x)\, dx = \int \frac{f(x)}{u'(x)}\, du\,.$$

Since $u = u(x)$ we also have implicitly $x = x(u)$, which converts, in principle, the last integral above into one only involving u. In practice, algebraic simplification may be used to simplify the result.

Example: Consider

$$\int \sec x \, dx = \int \frac{1}{\cos x} \, dx \, .$$

A bit of trial and error shows that the substitution $u = \sin x$ isn't one that we gets rejected right away. Since $du = \cos x \, dx$, we have

$$\int \frac{1}{\cos x} \, dx = \int \frac{1}{\cos^2 x} \, du \, .$$

Finally, since $\cos^2 x = 1 - \sin^2 x = 1 - u^2$, the above becomes

$$\int \sec x \, dx = \int \frac{1}{1 - u^2} \, du \, .$$

The last integral can be done with partial fractions (more about this later).

2.3. Algebraic methods

In the above algebraic simplification (really, trigonometric simplification) was used to do the final step of the substitution. Since such types of algebraic simplification are many and varied, it is hard to delineate them all. One that comes up repeatedly, however, is *completing the square.*

Generally speaking, there are often multiple ways to integrate the same function. When there are, the answers must, of course, be equivalent. Such answers may not appear to be at first glance, however, and it may require additional algebra to show that two results obtained by different methods are really the same.

2.4. Integration by parts

This is based on the differential form of the product rule, $d(uv) = u\,dv + v\,du$. In its integral form, this is

$$uv = \int u \, dv + \int v \, du \, ,$$

or, equivalently

$$\int u \, dv = uv - \int v \, du \, .$$

This allows us to break one integral up into pieces and turn it into another which hopefully is easier.

Note it is possible that after doing this one ends up with the same integral again. In this case, one can solve for the result algebraically.

Example: Consider $I = \int \sqrt{1-z^2}\,dz$. Letting $u = \sqrt{1-z^2}$ and $dv = dz$ we get $du = -z\,dz/sqrt1-z^2$ and $v = z$. Then

$$I = \int \sqrt{1-z^2}\,dz = z\sqrt{1-z^2} + \int \frac{z^2}{\sqrt{1-z^2}}\,dz\,.$$

The last term we simplify with algebra:

$$\int \frac{z^2}{\sqrt{1-z^2}}\,dz = \int \frac{z^2-1+1}{\sqrt{1-z^2}}\,dz = -\int \sqrt{1-z^2}\,dz + \int \frac{dz}{\sqrt{1-z^2}} = -I + \sin^{-1} z + C\,.$$

Therefore, we have

$$I = z\sqrt{1-z^2} - I + \sin^{-1} z + C \quad \Rightarrow \quad 2I = z\sqrt{1-z^2} + \sin^{-1} z + C$$

and finally

$$\int \sqrt{1-z^2}\,dz = \frac{1}{2} z\sqrt{1-z^2} + \frac{1}{2}\sin^{-1} z + \frac{C}{2}\,.$$

Since the constant C is arbitrary, the factor of 2 can be omitted if desired.

2.5. Integrals with powers and products of trig functions

Consider integrals of the form

$$\int sin^m x \cos^n x\,dx$$

where m and n are integers.

If m is odd, one can pull off one of the sines, leaving an even number. This even number of sines we convert to cosines using the identity $\sin^2 x = 1-\cos^2 x$ (note this is just $\sin^2 x + \cos^2 x = 1$ solved for $sin^2 x$). Then by making the substitution $u = \cos x$ the result will just involve integral powers of u.

Similarly, if n is odd, one pulls off one of the cosines, converts the rest to sines, and uses the substitution $u = \sin x$.

If both m and n are even, then one can make use of the trigonometric identities

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)\,,$$

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)\,,$$

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

to reduce the powers of the trig functions until one of the previous methods can deal with it.

Note also that by using integration by parts one can derive a *reduction formula* for integrals of even powers of sines and cosines, e.g.,

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx\,.$$

The above can be repeated for integrals of the form

$$\int \sec^n x\tan^m x\,dx \quad \text{and} \quad \int \csc^n x\cot^m x\,dx\,.$$

For the first integral, the trig identity we will use is $\tan^2 x + 1 = \sec^2 x$. (This follows from $\sin^2 x + \cos^2 x = 1$ after dividing both sides of the equation by $\cos^2 x$.) In addition, the substitution to be used is either

$$u = \tan x \quad \Rightarrow \quad du = \sec^2 x\,dx$$

or

$$u = \sec^2 x \quad \Rightarrow \quad du = \sec x\tan x\,dx\,.$$

Note that to use the first substitution we need two secants, and to use the second we need one secant and one tangent. Thus, if m is even, we pull off two secants, convert the rest to tangents, and use the first substitution, $u = \tan x$. If n is odd, we pull off one tangent and one secant, convert the rest of the tangents to secants, and use the second substitution $u = \sec x$. Note that for this to work at least one secant is needed; if there are none, other methods must be used. For example, with a little bit of trigonometric algebra, one can derive the reduction formulae

$$\int \tan^m x\,dx = \frac{1}{m-1}\tan^{m-1}x - \int \tan^{m-2}x\,dx\,,$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2}x\tan x + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx\,.$$

Finally, if m is even and secants are present, one can convert all of the tangents to secants, giving a series of integrals all involving powers of secants, and the above reduction formula can be used.

Note that if the powers are large the resulting algebra can be fairly extensive.

The other type of integral that can come up involves products of trig functions with different arguments, e.g. $\int \sin 3x \cos x \, dx$. In such cases, use of one or more of the identities

$$\sin a \cos b = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right],$$

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right],$$

$$\cos a \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right],$$

will simplify the integrand. It's hard to remember these formulae, but they can be derived from the more standard ones

$$\cos(a+b) = \cos a \cos b - \sin a \sin b,$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b.$$

2.6. Trigonometric substitutions

Integrals involving the expressions $(a^2 - x^2)^{1/2}$, $(a^2 + x^2)^{1/2}$ and $(x^2 - a^2)^{1/2}$ may be converted into a form looking like one of the integrals above by using an appropriate *trig substitution*. In each case, the appropriate substitution can be gleaned by considering the identity $\sin^2\theta + \cos^2\theta = 1$, or one of the two alternate identities which can be derived from this by dividing by either $\cos^2\theta$ or $\sin^2\theta$, i.e., $\tan^2\theta + 1 = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$. In each case, the goal is to make the expression inside the square root match up with an appropriate trig identity, with x one of the trig functions, so that the expression inside the square root is a perfect square.

For example, if one has $(a^2 - x^2)^{1/2}$, one can rewrite $\sin^2\theta + \cos^2\theta = 1$ as $1 - \sin^2\theta = \cos^2\theta$. In addition, if one multiplies by a^2, one has $a^2 - a^2\sin^2\theta = a^2\cos^2\theta$. In this case, one identifies x with $a\sin\theta$, so the substitution is $x = a\sin\theta$. Thus, $(a^2 - x^2)^{1/2}$ becomes $(a^2\cos^2\theta)^{1/2} = a|\cos\theta|$ and the square root disappears. Note in the result one must be careful of the range of θ and the *sign* of the cosine. Also, note that at the beginning one could have written instead $1 - \cos^2\theta = \sin^2\theta$, which would have led to the alternative substitution $x = a\cos\theta$. The final answer, written in terms of the original variable x, of course, must be the same no matter what choice is taken.

In summary,

1. With $x = a\sin\theta$, one gets $(a^2 - x^2)^{1/2} = a\cos\theta$, [or with $x = a\cos\theta$, $(a^2 - x^2)^{1/2} = a\sin\theta$].
2. With $x = a\tan\theta$, one gets $(a^2 + x^2)^{1/2} = a\sec\theta$.
3. With $x = a\sec\theta$, one gets $(x^2 - a^2)^{1/2} = a\tan\theta$.

For the latter two, $x = a\cot\theta$ and $x = a\csc\theta$ could also be used.

There are additional, more advanced ways for doing such integrals using *hyperbolic trig functions*. The basic definitions are

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}\,.$$

These are called the hyperbolic sine and cosine. Note that

$$\frac{d}{dx}\sinh x = \cosh x \quad \text{and} \quad \frac{d}{dx}\cosh x = \sinh x\,.$$

In addition, there is a hyperbolic tangent, secant, etc.,

$$\tanh x = \frac{\sinh x}{\cosh x} \quad \text{and} \quad \operatorname{sech} x = \frac{1}{\cosh x} \quad \text{etc.}$$

Furthermore, one can easily verify the identities $\cosh^2 x - \sinh^2 x = 1$ and $1 - \tanh^2 x = \operatorname{sech}^2 x$. Thus, we also have:

1. With $x = a\tanh\theta$, one gets $(a^2 - x^2)^{1/2} = a\operatorname{sech}\theta$.
2. With $x = a\sinh\theta$, one gets $(a^2 + x^2)^{1/2} = a\cosh\theta$.
3. With $x = a\cosh\theta$, one gets $(x^2 - a^2)^{1/2} = a\sinh\theta$.

Sometimes the hyperbolic trig substitutions produce integrals that are easier to deal with than the regular trig functions. The price, of course, is that one has to use these less familiar functions.

2.7. Partial fractions

This is an algebraic method for simplifying proper rational fractions (i.e., the numerator and denominator are polynomials, with the degree of the numerator less than that of the denominator) to a point where it can be integrated. The main idea is to reverse the process of putting terms over a common denominator. This is one of the few cases where specific steps to follow can be given. The difficult cases don't come up all that often, but they are included here for completeness.

Step 1: If the rational function is not proper, make it so by dividing the bottom into the top.

Step 2: Separate the denominator into its linear and quadratic factors (i.e., factor the denominator). [Note: linear factors are relatively straightforward, but in general one needs to be able to factor over quadratics, e.g., $x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$.]

Step 3: Reverse the process of putting things over a common denominator and separate the rational function into a sum of terms known as its *partial fraction expansion*.

Every linear factor $(x-\alpha_i)$ generates a term in the sum

$$\frac{A_i}{x-\alpha_i}, \quad \text{where } A_i \text{ is a constant}.$$

Every quadratic factor $(x^2+\beta_j x+\gamma_j)$ generates a term in the sum

$$\frac{B_j x + C_j}{x^2+\beta_j x+\gamma_j}, \quad \text{where } B_j \text{ and } C_j \text{ are constants.}$$

A repeated linear factor $(x-\alpha)^k$ generates in the sum the terms

$$\frac{A_1}{(x-\alpha)} + \frac{A_2}{(x-\alpha)^2} + \ldots + \frac{A_k}{(x-\alpha)^k}.$$

A repeated quadratic factor $(x^2+\beta x+\gamma)^n$ generates in the sum the terms

$$\frac{B_1 x + C_1}{(x^2+\beta x+\gamma)} + \frac{B_2 x + C_2}{(x^2+\beta x+\gamma)^2} + \ldots + \frac{B_n x + C_n}{(x^2+\beta x+\gamma)^n}.$$

One writes down all of the terms, puts things over a common denominator, determines the system of equations for the unknown coefficients A, B and C, and solves for them. If the factors are linear this process can be sped up by multiplying by the common denominator and evaluating successively at each of the roots.

Step 4: Integrate the various terms that have been produced. The terms

$$\int \frac{A\,dx}{x-\alpha} \quad \text{and} \quad \int \frac{A\,dx}{(x-\alpha)^m}$$

are easy, of course. Terms like

$$\int \frac{Bx+C}{x^2+\beta x+\gamma}\,dx$$

can be done by splitting the integrand into two terms, the first of which has the numerator the derivative of the denominator and the second of which is the remainder. The first term is integrated with a simple substitution and the second is done by completing the square in the denominator. Finally, terms like

$$\int \frac{Bx+C}{(x^2+\beta x+\gamma)^k}\,dx$$

can be done by splitting the integrand into two terms as before but this time using a reduction formula on the second term after completing the square, e.g.,

$$\int \frac{du}{(u^2+a^2)^k} = \frac{u}{2a^2(k-1)(u^2+a^2)^{k-1}} + \frac{2k-3}{2a^2(k-1)} \int \frac{du}{(u^2+a^2)^{k-1}}.$$

2.8. Rational functions of $\sin x$ and $\cos x$

The integral of any rational function of $\sin x$ and $\cos x$, for example,

$$\int \frac{dx}{a + b\cos x}\,,$$

can always be transformed into a rational function of a new variable z using the substitution $z = \tan(x/2)$. Since $1 + \tan^2(x/2) = \sec^2(x/2)$, we have

$$\cos^2(x/2) = \frac{1}{1+z^2} \quad \Rightarrow \quad \cos(x/2) = \frac{1}{\sqrt{1+z^2}}$$

and

$$\sin^2(x/2) = 1 - \cos^2(x/2) = \frac{z^2}{1+z^2} \quad \Rightarrow \quad \sin(x/2) = \frac{z}{\sqrt{1+z^2}}\,.$$

Then

$$\sin x = 2\sin(x/2)\cos(x/2) = \frac{2z}{1+z^2}\,,$$

$$\cos x = \cos^2(x/2) - \sin^2(x/2) = \frac{1-z^2}{1+z^2}\,,$$

$$dz = \frac{1}{2}\sec^2(x/2)\,dx = \frac{1}{2}\left[1 + \tan^2(x/2)\right]dx \quad \Rightarrow \quad dx = \frac{2\,dz}{1+z^2}\,.$$

Once the details of the substitution has been made, the rest of the integration typically follows other methods, such as partial fractions.